

A new variational formulation based on discontinuous Galerkin technique for a reaction-diffusion problem[☆]

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Abstract

In this paper, a new variational formulation based on discontinuous Galerkin technique for a reaction-diffusion problem is introduced, and the discontinuous Galerkin technique of this work is different from the general discontinuous Galerkin methods. The well posedness of the new formulation is given. Finally, it is pointed that the new variational formulation will be helpful to design better hybrid numerical methods which will not only strongly stable in spatial variable and absolutely stable in temporal variable but also be optimally convergent.

Keywords: Discontinuous Galerkin technique, variational formulation, inf-sup condition.

1. Introduction

In this work, we propose a new variational formulation based on discontinuous Galerkin technique for a reaction-diffusion problem within a new

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function space setting, the reaction-diffusion problem is

$$-\nabla \cdot (K(x)\nabla u) + u = f, \text{ in } \Omega, \quad (1.1)$$

$$u = 0, \text{ on } \partial\Omega, \quad (1.2)$$

where f is a real-valued function in $L^2(\Omega)$ and $0 < K_0 \leq K(x) \leq K_1$.

The problem (1.1)-(1.2) is an important and basic mathematical model, widely used in many fields. As for the theoretical result of the above model, one can see [1] and so on.

The first discontinuous Galerkin (DG) method for hyperbolic equations was introduced by [2], and since that time there has been an active development of DG methods for hyperbolic and nearly hyperbolic problems, resulting in a variety of different methods. For elliptic and parabolic equations, discontinuous finite elements were proposed by many researchers, such as [3–16].

The key idea of the paper is to propose a new variational formulation based on discontinuous Galerkin technique, and the discontinuous Galerkin technique is different from the general discontinuous Galerkin methods. The formulation satisfies a local conservation property, and we prove well posedness of the new formulation by proving and using inf-sup condition.

The paper is organized as follows. In Section 2, we introduce the new weak formulation of the problem (1.1)-(1.2). In Section 3, we investigate the well posedness of the variational formulation, which includes the continuity property of the bilinear form and the inf-sup condition. Finally, some concluding remarks are summarized.

2. A new variational formulation

Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain with Lipschitz boundary $\partial\Omega$ and let $\{P_h\}$ be a family of regular partitions of Ω into open elements E such that $\Omega = \text{int}(\bigcup_{E \in P_h} \bar{E})$.

The following notations will be used in our further considerations. Denote $h = \max_{E \in P_h} h_E$, where $h_E = \text{diam}(E)$. The set of all edges of the partition P_h is given by $\varepsilon_h = \{\gamma_k\}, k = 1, \dots, N_{edge}$, where N_{edge} represents the number of edges in the partition P_h . The interior interface Γ_{int} is then defined as the union of all common edges shared by elements of partition P_h , that is,

$$\Gamma_{\text{int}} = \bigcup_{k=1}^{N_{edge}} \gamma_k \setminus \partial\Omega.$$

For the sake of clarity in the notation, the jump and average operators are defined by

$$[v] = v|_{\gamma_e \subset \partial E_i} - v|_{\gamma_e \subset \partial E_j}, \langle v \rangle = \frac{1}{2}(v|_{\gamma_e \subset \partial E_i} + v|_{\gamma_e \subset \partial E_j}), \quad i > j, \quad (2.1)$$

where $\gamma_e = \text{int}(\partial E_i \cap \partial E_j)$ is the common edge in 2D (or interface in 3D) between two neighbouring elements, see Figure 1.

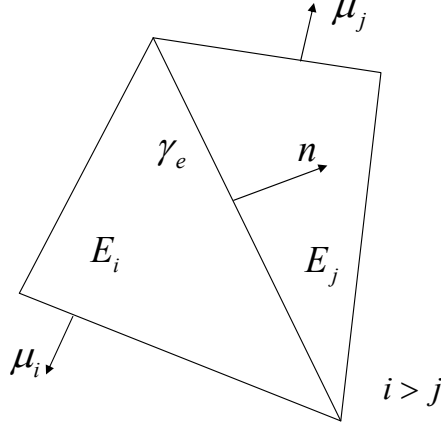


Figure 1: Geometrical definitions of neighbouring elements.

First, we introduce the following broken Sobolev space:

$$\mathcal{M}(P_h) = \{v \in L^2(\Omega) | v|_E \in H(\Delta, E), \forall E \in P_h, [\nabla v \cdot \mathbf{n}] \in L^2(\Gamma_{\text{int}})\},$$

where

$$H(\Delta, E) = \{v \in L^2(E) | \nabla \cdot \nabla v \in L^2(E)\} \subset H^1(E).$$

Notice here, that $v \in H(\Delta, E)$ implies $\nabla v \cdot \boldsymbol{\mu} \in H^{-1/2}(\partial\Omega)$. The norm $||| \cdot |||$ on $\mathcal{M}(P_h)$ is defined as

$$\begin{aligned} |||v|||^2 &= \sum_{E \in P_h} \left\{ \|v\|_*^2 + \frac{h^\nu}{p^\theta} \|K(x) \nabla v \cdot \boldsymbol{\mu}\|_{H^{-1/2}(\partial E)}^2 \right\} \\ &\quad + \sigma \frac{h^\lambda}{p^\zeta} \| [K(x) \nabla v \cdot \boldsymbol{\mu}] \|_{L^2(\Gamma_{\text{int}})}^2. \end{aligned} \quad (2.2)$$

where we denote that $\|v\|_*^2 = \int_E |K(x)| |\nabla v|^2 dx + \int_E |v|^2 dx = \|K^{\frac{1}{2}} \nabla v\|_{L^2(E)}^2 + \|v\|_{L^2(E)}^2$, and one can easily prove that norms $\|\cdot\|_*$ and $\|\cdot\|_{H^1(E)}$ are equivalent. The parameter $p \in \mathbb{R}$ that is introduced here represents the minimum

of all of the local orders of polynomial approximations p_E in the partition P_h . The parameters $\nu, \lambda, \theta, \zeta$ are greater than or equal to zero and that the subsequent norms in (2.2) are defined as

$$\|u\|_{H^{-1/2}(\partial E)} = \sup_{\varphi \in H^{1/2}(\partial E)} \frac{|\langle u, \varphi \rangle_{-1/2 \times 1/2, \partial E}|}{\|\varphi\|_{H^{1/2}(\partial E)}}, \quad (2.3)$$

$$\|\varphi\|_{H^{1/2}(\partial E)} = \inf_{\substack{w \in H^1(E) \\ \gamma_0 w = \varphi}} \|w\|_*, \quad (2.4)$$

where $\langle \cdot, \cdot \rangle_{-1/2 \times 1/2, \partial E}$ denotes the duality pairing in $H^{-1/2}(\partial E) \times H^{1/2}(\partial E)$, namely,

$$\langle u, v \rangle_{-1/2 \times 1/2, \partial E} = \int_{\partial E} u v ds. \quad (2.5)$$

And γ_0 denotes the trace operator

$$\gamma_0 : H^1(E) \rightarrow H^{1/2}(\partial E).$$

Now, the choice for the space of test functions, V , is the completion of $\mathcal{M}(P_h)$ with respect to the norm $||| \cdot |||$.

The new discontinuous variational formulation, within this new function space setting, is then stated as follows:

$$\text{Find } u \in V, \text{ s.t., } B(u, v) = L(v), \quad \forall v \in V, \quad (2.6)$$

where the bilinear form $B(u, v)$ and linear form $L(v)$ are defined as

$$\begin{aligned}
B(u, v) = & \sum_{E \in P_h} \left\{ \int_E (K(x) \nabla u \cdot \nabla v + uv) dx \right. \\
& - \int_{\partial E} (v(K(x) \nabla u \cdot \boldsymbol{\mu}) - (K(x) \nabla v \cdot \boldsymbol{\mu})u) ds \Big\} \\
& + \int_{\Gamma_{\text{int}}} (\langle v \rangle [K(x) \nabla u \cdot \boldsymbol{n}] - \langle u \rangle [K(x) \nabla v \cdot \boldsymbol{n}]) ds \\
& + \int_{\Gamma_{\text{int}}} \sigma \frac{h^\lambda}{p^\zeta} [K(x) \nabla u \cdot \boldsymbol{n}] [K(x) \nabla v \cdot \boldsymbol{n}] ds, \tag{2.7}
\end{aligned}$$

$$L(v) = \int_{\Omega} f v dx. \tag{2.8}$$

The formulation (2.6) (or VBVP (2.6)) is closely related to the DG method formulation by Oden, Babuska and Baumann [17]. In fact, choosing the subspace $\tilde{V}(P_h)$ of V of function with fluxes $\nabla v \cdot \boldsymbol{n} \in L^2(\partial E)$, and using the following identities:

$$\begin{aligned}
& \sum_{E \in P_h} \int_{\partial E} v(K(x) \nabla u \cdot \boldsymbol{\mu}) ds \\
& = \int_{\Gamma_{\text{int}}} [v(K(x) \nabla u \cdot \boldsymbol{n})] ds + \int_{\partial \Omega} v(K(x) \nabla u \cdot \boldsymbol{n}) ds, \tag{2.9}
\end{aligned}$$

and

$$[v(K(x) \nabla u \cdot \boldsymbol{n})] = \langle K(x) \nabla u \cdot \boldsymbol{n} \rangle [v] + \langle v \rangle [K(x) \nabla u \cdot \boldsymbol{n}], \tag{2.10}$$

we can get the DG formulation of [17]. The only difference would then be the addition of the last term in (2.7). This term has been incorporated in [18, 19], where it is accompanied by the jumps of the function $[v]$ across the element interfaces. We replace the $[v]$ jumps by the $[\nabla v \cdot \boldsymbol{\mu}]$ jumps, in order to prove both continuity and Inf-Sup properties of the bilinear form with respect to the space V , in which the norm is defined as $||| \cdot |||$.

3. Well posedness of the new variational formulation

In this section, we establish the well posedness of the variational formulation (2.6). Thus, we show that the solution of the problem (1.1)-(1.2) is also a solution to the weak problem. And we prove the existence, uniqueness and the continuous dependence on the input data of the solution to the variational formulation (2.6). Essential in some of these proofs are the continuity inf-sup conditions of the bilinear form (2.7).

Now, we introduce an important lemma [16] as follows.

Lemma 3.1. *If $u \in H(\Delta, \Omega)$, then u and $(\nabla u \cdot \mathbf{n})$ are weakly continuous across the element interface Γ_{int} in the sense that*

$$\int_{\gamma_e} [u] \varphi ds = 0, \quad \int_{\gamma_e} [\nabla u \cdot \mathbf{n}] \varphi ds = 0, \quad (3.1)$$

where φ belongs to $D(\gamma_e)$, and $\gamma_e = \text{int}(\partial E_i \in \partial E_j) \subset \Gamma_{\text{int}}$.

Theorem 3.1. *Let u be the solution of the problem (1.1)-(1.2). Then u is a solution to the variational formulation (2.6) as well.*

Proof. If we restrict (1.1) to an element $E \in P_h$, multiply this local equation by a test function $\varphi_E \in H^2(E)$, integrate over the element E , and apply Green's identity, we get

$$\int_E (K(x) \nabla u \cdot \nabla \varphi_E + u \varphi_E) dx - \int_{\partial E} \varphi_E (K(x) \nabla u \cdot \boldsymbol{\mu}) ds = \int_E f \varphi_E dx. \quad (3.2)$$

Repeating this for all $E \in P_h$, extending each φ_E to zero outside of E , and summing in E , yields

$$\begin{aligned} & \sum_{E \in P_h} \left\{ \int_E (K(x) \nabla u \cdot \nabla \varphi + u \varphi) dx - \int_{\partial E} \varphi (K(x) \nabla u \cdot \boldsymbol{\mu}) ds \right\} \\ &= \sum_{E \in P_h} \int_E f \varphi dx. \end{aligned} \quad (3.3)$$

where

$$\varphi = \sum_{E \in P_h} \varphi_E \in \{v \in L^2(\Omega) \mid v|_E \in H^2(E), \forall E \in P_h\}.$$

Since u is the solution of the problem (1.1)-(1.2), we know that u satisfies the Dirichlet boundary condition on $\partial\Omega$. In addition, it is known that u belongs to $H(\Delta, \Omega)$. From Lemma 3.1, we know that u and $(\nabla u \cdot \mathbf{n})$ are weakly continuous across the element interfaces to the variational formulation in a weak sense, which yields

$$\begin{aligned} & \sum_{E \in P_h} \left\{ \int_E (K(x) \nabla u \cdot \nabla \varphi + u \varphi) dx - \int_{\partial E} \varphi (K(x) \nabla u \cdot \boldsymbol{\mu}) ds \right\} \\ & + \int_{\Gamma_{\text{int}}} \langle \varphi \rangle [K(x) \nabla u \cdot \mathbf{n}] ds + \int_{\Gamma_{\text{int}}} [u] \langle K(x) \nabla \varphi \cdot \mathbf{n} \rangle ds \\ & + \int_{\partial\Omega} u (K(x) \nabla \varphi \cdot \boldsymbol{\mu}) ds + \int_{\Gamma_{\text{int}}} \sigma \frac{h^\lambda}{p^\zeta} [K(x) \nabla u \cdot \mathbf{n}] [K(x) \nabla \varphi \cdot \mathbf{n}] ds \\ & = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in H^2(P_h). \end{aligned} \tag{3.4}$$

Combining (2.9) and (2.10) gives

$$\begin{aligned} & \sum_{E \in P_h} \left\{ \int_E (K(x) \nabla u \cdot \nabla \varphi + u \varphi) dx \right. \\ & \left. - \int_{\partial E} (\varphi (K(x) \nabla u \cdot \boldsymbol{\mu}) - u (K(x) \nabla \varphi \cdot \boldsymbol{\mu})) ds \right\} \\ & + \int_{\Gamma_{\text{int}}} (\langle \varphi \rangle [K(x) \nabla u \cdot \mathbf{n}] - \langle u \rangle [K(x) \nabla \varphi \cdot \mathbf{n}]) ds \\ & + \int_{\Gamma_{\text{int}}} \sigma \frac{h^\lambda}{p^\zeta} [K(x) \nabla u \cdot \mathbf{n}] [K(x) \nabla \varphi \cdot \mathbf{n}] ds, \\ & = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in H^2(P_h). \end{aligned} \tag{3.5}$$

Applying the density of $H^2(P_h)$ in V , we complete the proof. \square

3.1. Continuity property

Theorem 3.2. *Let $B(\cdot, \cdot)$ be the bilinear form as defined in (2.7). If $\sigma > 0$, then there exists $M > 0$ such that*

$$|B(u, v)| \leq M \|u\| \|v\|, \quad \forall u, v \in V, \quad (3.6)$$

Proof. By the definition of the average, we can obtain

$$\int_{\Gamma_{\text{int}}} \langle v \rangle [K(x) \nabla u \cdot \mathbf{n}] ds = \frac{1}{2} \sum_{E \in P_E} \int_{\partial E \cap \Gamma_{\text{int}}} v [K(x) \nabla u \cdot \mathbf{n}] ds, \quad u, v \in V. \quad (3.7)$$

And by the definition of $B(\cdot, \cdot)$, then we get

$$\begin{aligned} B(u, v) &= \sum_{E \in P_h} \left\{ \int_E (K(x) \nabla u \cdot \nabla v + uv) dx \right. \\ &\quad - \int_{\partial E} v (K(x) \nabla u \cdot \boldsymbol{\mu}) ds + \int_{\partial E} u (K(x) \nabla v \cdot \boldsymbol{\mu}) ds \\ &\quad + \frac{1}{2} \int_{\partial E \cap \Gamma_{\text{int}}} v [K(x) \nabla u \cdot \mathbf{n}] ds - \frac{1}{2} \int_{\partial E \cap \Gamma_{\text{int}}} u [K(x) \nabla v \cdot \mathbf{n}] ds \left. \right\} \\ &\quad + \sigma \frac{h^\lambda}{p^\zeta} \int_{\Gamma_{\text{int}}} [K(x) \nabla u \cdot \mathbf{n}] [K(x) \nabla v \cdot \mathbf{n}] ds. \end{aligned} \quad (3.8)$$

Applying the Schwarz inequality, (2.3) and (2.4), we get

$$\begin{aligned} B(u, v) &\leq \max \left\{ 3, \frac{Cp^\theta}{h^\nu}, \frac{Cp^\zeta}{4\sigma h^\lambda} + 1 \right\} \\ &\cdot \left\{ \sum_{E \in P_h} \left(\|u\|_*^2 + \frac{h^\nu}{p^\theta} \|K(x) \nabla u \cdot \boldsymbol{\mu}\|_{H^{-1/2}(\partial E)}^2 \right) \right. \\ &\quad \left. + \sigma \frac{h^\lambda}{p^\zeta} \| [K(x) \nabla u \cdot \mathbf{n}] \|_{L^2(\Gamma_{\text{int}})}^2 \right\}^{1/2} \\ &\cdot \left\{ \sum_{E \in P_h} \left(\|v\|_*^2 + \frac{h^\nu}{p^\theta} \|K(x) \nabla v \cdot \boldsymbol{\mu}\|_{H^{-1/2}(\partial E)}^2 \right) \right. \\ &\quad \left. + \sigma \frac{h^\lambda}{p^\zeta} \| [K(x) \nabla v \cdot \mathbf{n}] \|_{L^2(\Gamma_{\text{int}})}^2 \right\}^{1/2}. \end{aligned}$$

Denote $M = \max \left\{ 3, \frac{Cp^\theta}{h^\nu}, \frac{Cp^\zeta}{4\sigma h^\lambda} + 1 \right\}$, where $C = \max\{\frac{1}{K_0}, 1\}$, hence the proof is completed. \square

3.2. The Inf-sup condition

3.2.1. The auxiliary problems

Given an arbitrary $u \in V$, find for every $E \in P_h$ the function z_E , such that

$$\begin{aligned} -\nabla \cdot (K(x)\nabla z_E) + z_E &= 0, & \text{in } E, \\ K(x)\nabla z_E \cdot \boldsymbol{\mu} &= K(x)\nabla u \cdot \boldsymbol{\mu}, & \text{on } \partial E. \end{aligned} \quad (3.9)$$

The equivalent variational formulation of (3.9) is that given $u \in V$, find $z_E \in H^1(E)$ such that

$$(z_E, v)_{*,E} = \int_{\partial E} (K(x)\nabla u \cdot \boldsymbol{\mu}) \gamma_0 v ds, \quad v \in H^1(E), \quad (3.10)$$

where $(\cdot, \cdot)_{*,E}$ denotes the inner product in $H^1(E)$, and it can be proved that $(v, v)_{*,E} = \|v\|_*^2$ easily. By the generalized Lax-Milgram theorem it follows that the problem (3.10) has a unique solution $z_E \in H^1(E)$.

Remark 3.1. *Substituting z_E and u for v in (3.10), we obtain the following two identities:*

$$\begin{aligned} \|z_E\|_*^2 &= \int_{\partial E} (K(x)\nabla u \cdot \boldsymbol{\mu}) \gamma_0 z_E ds, \\ (z_E, u)_{*,E} &= \int_{\partial E} (K(x)\nabla u \cdot \boldsymbol{\mu}) \gamma_0 u ds. \end{aligned}$$

From [16, 20], we know that the following result holds.

Theorem 3.3. *Given $u \in V$, let $z_E = z_E(u)$ be the unique solution to (3.10), then the following relation holds:*

$$\|z_E\|_* = \|K(x)\nabla u \cdot \boldsymbol{\mu}\|_{H^{-1/2}(\partial E)}.$$

3.2.2. Inf-sup condition on the space V

In this section, we prove that bilinear form $B(\cdot, \cdot)$ satisfies the Inf-Sup condition with respect to the norm $||| \cdot |||$, defined by (2.2). Let us introduce the extension operator $\Psi_E : H^1(E) \rightarrow V$,

$$\Psi_E(v_E) = \begin{cases} v_E, & \text{in } E, \\ 0, & \text{in } \Omega \setminus E. \end{cases}$$

Hence, given a function $u \in V$, we can solve (3.10) for a set of functions $z_E(u)$ and construct a function $\hat{u} \in V$, such that

$$\hat{u} = u + \beta \sum_{E \in P_h} \Psi_E(z_E), \quad (3.11)$$

where $\beta \in \mathbb{R}$.

Lemma 3.2. *Given $u \in V$, then for every $\beta \in \mathbb{R}$ there exists a strictly positive $\xi_1 = \xi_1(h, p)$ such that*

$$|||\hat{u}||| \leq \xi_1 |||u|||.$$

Proof. Substitution of the definition of \hat{u} into (2.2) and recalling from (3.9) that $K(x)\nabla z_E \cdot \boldsymbol{\mu} = K(x)\nabla u \cdot \boldsymbol{\mu}$ on ∂E , we obtain

$$\begin{aligned} |||\hat{u}|||^2 &= \sum_{E \in P_h} \left\{ \|u\|_*^2 + 2(u, \beta z_E)_{*,E} + \|\beta z_E\|_*^2 \right. \\ &\quad \left. + (1 + \beta)^2 \frac{h^\nu}{p^\theta} \|K(x)\nabla u \cdot \boldsymbol{\mu}\|_{H^{-1/2}(\partial E)}^2 \right\} \\ &\quad + \sigma \frac{h^\lambda}{p^\zeta} (1 + \beta)^2 \| [K(x)\nabla u \cdot \boldsymbol{n}] \|_{L^2(\Gamma_{\text{int}})}^2. \end{aligned}$$

Using the Schwarz inequality, triangle inequality, Theorem 3.3, and Young's inequality as follows:

$$2(u, z_E)_{*,E} \leq \varepsilon \|u\|_*^2 + \frac{1}{\varepsilon} \|z_E\|_*^2, \quad \varepsilon > 0, \quad (3.12)$$

we obtain (here taking $\varepsilon = 1$)

$$\begin{aligned} |||\hat{u}|||^2 &\leq \sum_{E \in P_h} \left\{ 2\|u\|_*^2 + \left((1 + \beta)^2 + \frac{2\beta^2 p^\theta}{h^\nu} \right) \frac{h^\nu}{p^\theta} \|K(x) \nabla u \cdot \boldsymbol{\mu}\|_{H^{-1/2}(\partial E)}^2 \right\} \\ &\quad + (1 + \beta)^2 \sigma \frac{h^\lambda}{p^\zeta} \| [K(x) \nabla u \cdot \mathbf{n}] \|_{L^2(\Gamma_{\text{int}})}^2. \end{aligned}$$

Thus, the assertion holds with

$$\xi_1 = \sqrt{\max \left\{ 2, (1 + \beta)^2 + \frac{2\beta^2 p^\theta}{h^\nu} \right\}}.$$

□

Lemma 3.3. *Given $u \in V$, then there exists $\xi_2 = \xi_2(\sigma, h, p) > 0$ such that*

$$B(u, \hat{u}) \geq \xi_2 |||u|||^2. \quad (3.13)$$

Proof. By replacing v by \hat{u} in the definition of $B(u, v)$, and recalling that $K(x) \nabla z_E \cdot \boldsymbol{\mu} = K(x) \nabla u \cdot \boldsymbol{\mu}$, we get

$$\begin{aligned} B(u, \hat{u}) &= \sum_{E \in P_h} \left\{ \|u\|_*^2 + \beta(u, z_E)_{*,E} - \beta \int_{\partial E} z_E (K(x) \nabla u \cdot \boldsymbol{\mu}) ds \right. \\ &\quad \left. + \beta \int_{\partial E} u (K(x) \nabla u \cdot \boldsymbol{\mu}) ds \right\} + \beta \int_{\Gamma_{\text{int}}} \langle z_E \rangle [K(x) \nabla u \cdot \mathbf{n}] ds \\ &\quad - \beta \int_{\Gamma_{\text{int}}} \langle u \rangle [K(x) \nabla u \cdot \mathbf{n}] ds + \sigma \frac{h^\lambda}{p^\zeta} (1 + \beta) \| [K(x) \nabla u \cdot \mathbf{n}] \|_{L^2(\Gamma_{\text{int}})}^2. \end{aligned}$$

For simplicity, the traces $\gamma_0 z_E$ and $\gamma_0 u$ have been denoted as z_E and u , respectively. Now, using the identities given in Remark 3.1, we can rewrite the above expression as follows

$$\begin{aligned} B(u, \hat{u}) &= \sum_{E \in P_h} \left\{ \|u\|_*^2 + 2\beta(u, z_E)_{*,E} - \beta \|z_E\|_*^2 \right\} \\ &\quad + \beta \int_{\Gamma_{\text{int}}} \langle z_E \rangle [K(x) \nabla u \cdot \mathbf{n}] ds - \beta \int_{\Gamma_{\text{int}}} \langle u \rangle [K(x) \nabla u \cdot \mathbf{n}] ds \\ &\quad + \sigma \frac{h^\lambda}{p^\zeta} (1 + \beta) \| [K(x) \nabla u \cdot \mathbf{n}] \|_{L^2(\Gamma_{\text{int}})}^2. \end{aligned} \quad (3.14)$$

If we take a closer look at the terms involving integrals over Γ_{int} , we see that

$$\begin{aligned}\beta \int_{\Gamma_{\text{int}}} \langle z_E \rangle [K(x) \nabla u \cdot \mathbf{n}] ds &\geq -\frac{|\beta|}{4} \sum_{E \in P_h} \|z_E\|_*^2 - \frac{|\beta|}{2} \| [K(x) \nabla u \cdot \mathbf{n}] \|_{L^2(\Gamma_{\text{int}})}^2, \\ -\beta \int_{\Gamma_{\text{int}}} \langle u \rangle [K(x) \nabla u \cdot \mathbf{n}] ds &\geq -\frac{|\beta|}{4} \sum_{E \in P_h} \|u\|_*^2 - \frac{|\beta|}{2} \| [K(x) \nabla u \cdot \mathbf{n}] \|_{L^2(\Gamma_{\text{int}})}^2.\end{aligned}$$

Now, back substitution of these two results into (3.14), yields

$$\begin{aligned}B(u, \hat{u}) &\geq \sum_{E \in P_h} \left\{ \left(1 - \frac{|\beta|}{4}\right) \|u\|_*^2 + 2\beta(u, z_E)_{*,E} - \left(\beta + \frac{|\beta|}{4}\right) \|z_E\|_*^2 \right\} \\ &\quad + \left(\sigma \frac{h^\lambda}{p^\zeta} (1 + \beta) - |\beta|\right) \| [K(x) \nabla u \cdot \mathbf{n}] \|_{L^2(\Gamma_{\text{int}})}^2.\end{aligned}$$

Using (3.12), Cauchy-Schwarz inequality and Theorem 3.3, we obtain

$$\begin{aligned}B(u, \hat{u}) &\geq \sum_{E \in P_h} \left\{ \left(1 - \varepsilon|\beta| - \frac{|\beta|}{4}\right) \|u\|_*^2 \right. \\ &\quad \left. - \left(\beta + \frac{|\beta|}{4} + \frac{|\beta|}{\varepsilon}\right) \|K(x) \nabla u \cdot \boldsymbol{\mu}\|_{H^{-1/2}(\partial E)}^2 \right\} \\ &\quad + \left(\sigma \frac{h^\lambda}{p^\zeta} (1 + \beta) - |\beta|\right) \| [K(x) \nabla u \cdot \mathbf{n}] \|_{L^2(\Gamma_{\text{int}})}^2.\end{aligned}$$

With $\beta < 0$, we have

$$\begin{aligned}B(u, \hat{u}) &\geq \sum_{E \in P_h} \left\{ \left(1 - \varepsilon|\beta| - \frac{|\beta|}{4}\right) \|u\|_*^2 \right. \\ &\quad \left. + \left(\frac{3|\beta|}{4} - \frac{|\beta|}{\varepsilon}\right) \|K(x) \nabla u \cdot \boldsymbol{\mu}\|_{H^{-1/2}(\partial E)}^2 \right\} \\ &\quad + \left(\sigma \frac{h^\lambda}{p^\zeta} (1 - |\beta|) - |\beta|\right) \| [K(x) \nabla u \cdot \mathbf{n}] \|_{L^2(\Gamma_{\text{int}})}^2.\end{aligned}$$

The second term in the right hand side is only positive for $\varepsilon > 4/3$. If we take $\varepsilon = 2$, then we get

$$B(u, \hat{u}) \geq \min \left\{ 1 - \frac{9}{4}|\beta|, \frac{|\beta|p^\theta}{4h^\nu}, 1 - |\beta| - \frac{|\beta|p^\zeta}{\sigma h^\lambda} \right\} \|u\|^2.$$

It is clear that, given the parameters $\sigma, \lambda, \zeta, \nu$ and θ , we can always find a coefficient β such that there exists a $\xi_2(\sigma, h, p)$, denoted by

$$\xi_2 = \min \left\{ 1 - \frac{9}{4}|\beta|, \frac{|\beta|p^\theta}{4h^\nu}, 1 - |\beta| - \frac{|\beta|p^\zeta}{\sigma h^\lambda} \right\}, \quad (3.15)$$

that satisfies the inequality (3.13). \square

Theorem 3.4. *Given $\sigma > 0$, there exists $\gamma = \gamma(\sigma, h, p) > 0$ such that*

$$\sup_{v \in V \setminus \{0\}} \frac{|B(u, v)|}{|||v|||} \geq \gamma |||u|||, \quad \forall u \in V. \quad (3.16)$$

Proof. By definition of the supremum, we can obtain

$$\sup_{v \in V \setminus \{0\}} \frac{|B(u, v)|}{|||v|||} \geq \frac{|B(u, \hat{u})|}{|||\hat{u}|||}, \quad \forall u \in V. \quad (3.17)$$

where \hat{u} is defined by (3.11). Next, by applying Lemmas 3.2 and 3.3, we obtain

$$\sup_{v \in V \setminus \{0\}} \frac{|B(u, v)|}{|||v|||} \geq \frac{|B(u, \hat{u})|}{|||\hat{u}|||} \geq \frac{\xi_2(\sigma, h, p)}{\xi_1(\sigma, h, p)} |||u|||, \quad \forall u \in V. \quad (3.18)$$

Taking $\gamma = \xi_2/\xi_1$, we finish the proof. \square

Corollary 3.1. *If $\lambda = \nu = \theta = \zeta = 0$, then the inf-sup coefficient γ is a constant.*

Proof. For simplicity, we set $\sigma = 1$. Choosing $\beta = 4/10$ and using (3.15), we have $\xi_1 = \sqrt{288}/10$ and $\xi_2 = 1/10$, respectively, and it follows that $\gamma = 1/\sqrt{288}$. \square

Corollary 3.2. *If $\lambda = \nu$ and $\theta = \zeta$, then for $h^\lambda/p^\zeta < 1$ the coefficient γ is bounded a constant $C > 0$.*

Proof. We still set that $\sigma = 1$, but now we choose $\beta = 4h^\nu/10p^\theta$. If we take $h^\lambda/p^\zeta < 1$, we obtain the following inequalities from (3.15)

$$\xi_1 \leq \frac{\sqrt{228}}{10}, \quad \xi_2 \geq \frac{1}{10}.$$

Hence, we conclude that $\gamma \geq 1/\sqrt{228}$. \square

3.3. Existence and uniqueness

Lemma 3.4. *If $f \in L^2(\Omega)$, then there exists a unique solution $w \in V$ to the VBVP (2.6) that is a solution to the problem (1.1)-(1.2).*

Proof. First, we introduce the classical variational formulation of the model problem (1.1)-(1.2) in $H_0^1(\Omega)$:

$$\text{Find } w \in H_0^1(\Omega), \text{ s.t., } A(w, v) = L(v), \quad \forall v \in H_0^1(\Omega), \quad (3.19)$$

where $L(v)$ is defined by (2.8) and the bilinear form $A(w, v) : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$ is defined by

$$A(w, v) = \int_{\Omega} (K(x) \nabla w \cdot \nabla v + wv) dx.$$

By the generalized Lax-Milgram theorem and by equivalence of this formulation to the problem (1.1)-(1.2), we know that if $f \in L^2(\Omega)$ there exists a unique solution $w \in H_0^1(\Omega) \cap H(\Delta, \Omega) \subset V$ to (3.19) that satisfies the model problem in a distributional sense. Consequently, by Theorem 3.1, we know that $w \in V$ is a solution to the VBVP (2.6) as well.

Thus, following the existence theory for the continuous variational formulation, we can easily prove the existence of a solution to the VBVP (2.6), and

we omit the detail of the proof. Also, we know that the solution is unique because the bilinear form $B(u, v)$ of (2.7) is positive definite, i.e.,

$$B(v, v) = \sum_{E \in P_h} \|v\|_*^2 + \sigma \frac{h^\lambda}{p^\eta} \|[K(x) \nabla v \cdot \mathbf{n}]\|_{L^2(\Gamma_{\text{int}})}^2 > 0, \quad \forall v \in V \setminus \{0\}.$$

□

3.4. Stability

The last requirement to ensure well posedness of the weak formulation is stability, i.e., the VBVP (2.6) is continuously dependent of the solution on the input data. Corresponding to Corollary 3.1 and 3.2, we have two propositions 3.1 and 3.2 as follows.

Proposition 3.1. *If $\sigma = 1, \lambda = \zeta = 0$, and $\nu = \theta = 0$, then the solution to the VBVP (2.6) depends continuously on the input data, i.e., given a small perturbation $\delta f \in L^2(\Omega)$, then there exists a unique perturbation $\delta u \in V$ such that*

$$\|\delta u\|_{H^1(P_h)} \leq |||\delta u||| \leq C \|\delta f\|_{L^2(\Omega)},$$

where C is a constant independent of h and p .

Proof. Let $\delta f \in L^2(\Omega)$ be a perturbation in the input data f . Consequently, since the problem under consideration is linear, this leads to a perturbation $\delta u \in V$ in the solution u , which satisfies

$$B(\delta u, v) = \int_{\Omega} \delta f v dx, \quad \forall v \in V. \quad (3.20)$$

Using Theorem 3.4, we have

$$|||\delta u||| \leq \frac{1}{\gamma} \sup_{v \in V \setminus \{0\}} \frac{|B(\delta u, v)|}{|||v|||}. \quad (3.21)$$

Using (3.20), (3.21) and the Cauchy-Schwarz inequality, we obtain

$$|||\delta u||| \leq \frac{1}{\gamma} \sup_{v \in V \setminus \{0\}} \frac{|\int_{\Omega} \delta f v dx|}{|||v|||} \leq \frac{1}{\gamma} \|\delta f\|_{L^2(\Omega)}. \quad (3.22)$$

If $\sigma = 1, \lambda = \nu = \theta = \zeta = 0$, using Lemmas 3.2 and 3.3, Theorem 3.4, Corollary 3.1 and (3.22), we can take $C = \frac{1}{\gamma} = \sqrt{228}$ to complete the proof. \square

Proposition 3.2. *If $\lambda = \nu$, $\theta = \zeta$, and $h^\lambda/p^\zeta < 1$, then the solution u to the VBVP (2.6) depends continuously on the input data, i.e., given a perturbation $\delta f \in L^2(\Omega)$, there exists a unique perturbation $\delta u \in V$ such that*

$$\|\delta u\|_{H^1(P_h)} \leq |||\delta u||| \leq C \|\delta f\|_{L^2(\Omega)},$$

where C is a constant independent of h and p .

Proof. Following the proof of Proposition 3.1, and using Corollary 3.2, we get

$$\|\delta u\|_{H^1(P_h)} \leq \frac{1}{\gamma} \|\delta f\|_{L^2(\Omega)}. \quad (3.23)$$

Therefore, the proof is completed. \square

4. Conclusion

We propose a new variational formulation by a new discontinuous Galerkin technique for a two-dimensional reaction-diffusion problem with Dirichlet boundary conditions. Our preliminary study discovers that there is a strong and intimate connection between the new variational formulation and discontinuous Galerkin methods for the reaction-diffusion problem with discontinuous coefficient, we intend to further explore this relation, in particular,

to make use of this relation to design better hybrid numerical methods which hopefully will not only strongly stable in spatial variable and absolutely stable in temporal variable but also be optimally convergent.

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